§ Local expressions of curvatures

Suppose we have a parametrization $X(u_1, u_2)$ of S.

We can express the 1st & 2nd f.f. of S as 2×2 symmetric matrices (under the basis $\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\}$ of T_pS)

$$(g_{ij}) = \begin{pmatrix} \langle \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_1} \rangle & \langle \frac{\partial X}{\partial u_1}, \frac{\partial X}{\partial u_2} \rangle \\ \langle \frac{\partial X}{\partial u_2}, \frac{\partial X}{\partial u_1} \rangle & \langle \frac{\partial X}{\partial u_2}, \frac{\partial X}{\partial u_2} \rangle \end{pmatrix}$$

$$(A_{ij}) = \begin{pmatrix} \langle \frac{\partial X}{\partial u_1}, -\frac{\partial N}{\partial u_1} \rangle & \langle \frac{\partial X}{\partial u_1}, -\frac{\partial N}{\partial u_2} \rangle \\ \langle \frac{\partial X}{\partial u_2}, -\frac{\partial N}{\partial u_1} \rangle & \langle \frac{\partial X}{\partial u_2}, -\frac{\partial N}{\partial u_2} \rangle \end{pmatrix}$$

Note: "Differentiate by part", we can also write

$$\boxed{\langle \frac{\partial X}{\partial u_i}, -\frac{\partial N}{\partial u_j} \rangle = \langle \frac{\partial^2 X}{\partial u_i \partial u_j}, N \rangle}$$

This is usually easier to calculate!

Question: Can we compute the curvatures K and H in terms of (g_{ij}) and (A_{ij})?

Yes!

Prop: Under the same basis of T_pS as above, the shape operator $S = -dN_p : T_pS \longrightarrow T_pS$ is given by the matrix:

$$\boxed{S = (g_{ij})^{-1}(A_{ij})} \quad \text{———} \quad (\#)$$

Proof: By definition,

$$\begin{cases} S\left(\frac{\partial}{\partial u_1}\right) = -dN\left(\frac{\partial}{\partial u_1}\right) = -\frac{\partial N}{\partial u_1} = a\frac{\partial}{\partial u_1} + b\frac{\partial}{\partial u_2} \\ S\left(\frac{\partial}{\partial u_2}\right) = -dN\left(\frac{\partial}{\partial u_2}\right) = -\frac{\partial N}{\partial u_2} = c\frac{\partial}{\partial u_1} + d\frac{\partial}{\partial u_2} \end{cases}$$

Written as matrices,

$$\begin{pmatrix} | & | \\ -\frac{\partial N}{\partial u_1} & -\frac{\partial N}{\partial u_2} \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} \\ | & | \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\Rightarrow \underbrace{\begin{pmatrix} - & \frac{\partial}{\partial u_1} & - \\ - & \frac{\partial}{\partial u_2} & - \end{pmatrix} \begin{pmatrix} | & | \\ -\frac{\partial N}{\partial u_1} & -\frac{\partial N}{\partial u_2} \\ | & | \end{pmatrix}}_{(A_{ij})} = \underbrace{\begin{pmatrix} - & \frac{\partial}{\partial u_1} & - \\ - & \frac{\partial}{\partial u_2} & - \end{pmatrix} \begin{pmatrix} | & | \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} \\ | & | \end{pmatrix}}_{(g_{ij})} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Multiplying $(g_{ij})^{-1}$ yields $(\#)$.

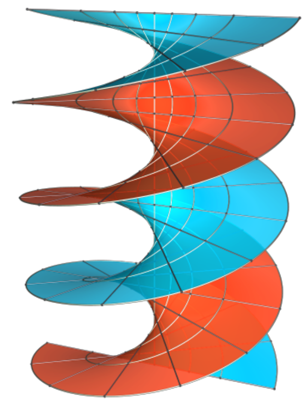
$\blacksquare$

Corollary:

$$\boxed{K = \frac{\det(A_{ij})}{\det(g_{ij})}} \quad \& \quad \boxed{H = \operatorname{tr}\left((g_{ij})^{-1}(A_{ij})\right)}$$

These formulas are very useful for computations!

Example: (Helicoid)



$$\mathbf{X}(u,v) = (v\cos u,\ v\sin u,\ u)$$

$$\frac{\partial \mathbf{X}}{\partial u} = (-v\sin u,\ v\cos u,\ 1)$$

$$\frac{\partial \mathbf{X}}{\partial v} = (\cos u,\ \sin u,\ 0)$$

1st f.f. $$(g_{ij}) = \begin{pmatrix} 1+v^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$N = \frac{\frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v}}{\left\| \frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} \right\|} = \frac{(-\sin u,\ \cos u,\ -v)}{\sqrt{1+v^2}}$$

$$\frac{\partial^2 \mathbf{X}}{\partial u^2} = (-v\cos u,\ -v\sin u,\ 0)$$

$$\frac{\partial^2 \mathbf{X}}{\partial u \partial v} = (-\sin u,\ \cos u,\ 0)$$

$$\frac{\partial^2 \mathbf{X}}{\partial v^2} = (0,0,0)$$

(Note: Compute $\frac{\partial N}{\partial v}$ is very complicated!)

2^{nd} f.f. $\quad (A_{ij}) = \begin{pmatrix} 0 & \frac{1}{\sqrt{1+v^2}} \\ \frac{1}{\sqrt{1+v^2}} & 0 \end{pmatrix}$

Hence, the shape operator is given by

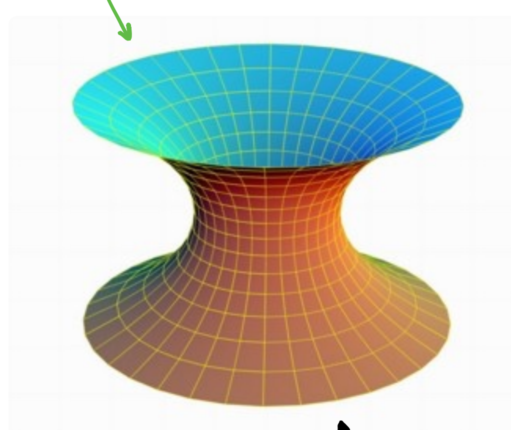
$$S = (g_{ij})^{-1}(A_{ij}) = \begin{pmatrix} 0 & \frac{1}{(1+v^2)^{3/2}} \\ \frac{1}{\sqrt{1+v^2}} & 0 \end{pmatrix}$$

$\boxed{K = \det S = -\frac{1}{(1+v^2)^2}}$ & $\boxed{H = \operatorname{tr} S \equiv 0}$

<u>Def$^{\underline{n}}$</u>: S is a minimal surface $\iff$ $H \equiv 0$.

<u>Examples</u>: plane, helicoid, catenoid (Ex.) ......

These are model for soap films which (locally) minimize area / energy.



$$X(u,v) = (\cosh v \cos u, \cosh v \sin u, v)$$

§ Local isometries and rigid motions

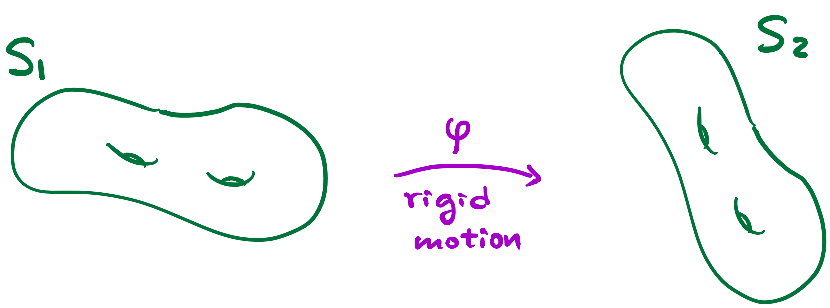
Recall: $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ rigid motion

$$\Longleftrightarrow \quad \langle d\varphi_p(v), d\varphi_p(w) \rangle = \langle v, w \rangle$$

$$\forall p \in \mathbb{R}^3, \quad \forall v, w \in \mathbb{R}^3 = T_p\mathbb{R}^3$$

$$\Longleftrightarrow \quad \varphi(x) = \underbrace{A}_{\text{rotation, reflection}} x + \underbrace{b}_{\text{translation}} \quad \text{where } A \in O(3),\ b \in \mathbb{R}^3$$

Def$^{\underline{n}}$: Two surfaces S_1, S_2 in $\mathbb{R}^3$ are congruent if $\exists$ rigid motion $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $\varphi(S_1) = S_2$.



Note: Congruent surfaces have the same geometry, both intrinsic and extrinsic!

intrinsic: depends only on 1st f.f

extrinsic: depends both on 1st & 2nd f.f.

Question: Is there a transformation that **ONLY** preserves the intrinsic geometry?

Def$^{\underline{n}}$: A smooth map $\varphi : S_1 \longrightarrow S_2$ between surfaces is said to be a local isometry if $\forall p \in S_1$,

$$d\varphi_p : T_pS_1 \longrightarrow T_{\varphi(p)}S_2 \quad \text{is a linear isometry}$$

i.e. $\langle d\varphi_p(v), d\varphi_p(w) \rangle = \langle v, w \rangle \quad \forall p \in S_1$
$\forall v, w \in T_pS_1$

If, furthermore, φ is a diffeomorphism, then we say that φ is an isometry.

Example: Any rigid motion $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ restricts to an isometry $\varphi|_S : S \to \varphi(S)$.

Def$^{\underline{n}}$: (1) Two surfaces S and S' are isometric if $\exists$ isometry $\varphi : S \to S'$.

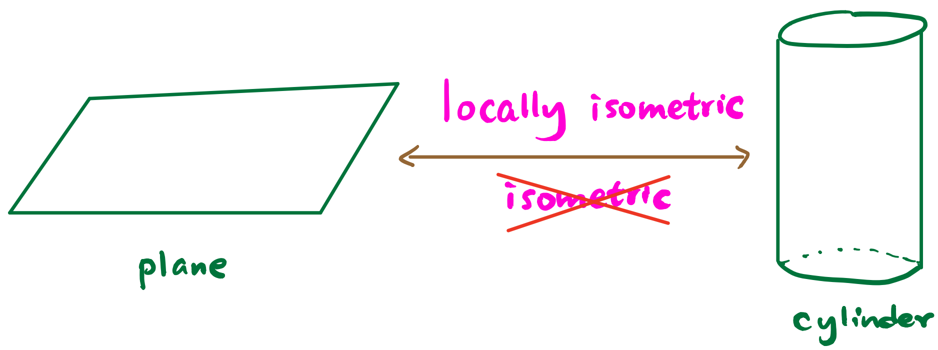
(2) Two surfaces S and S' are locally isometric if $\forall p \in S$, $\exists$ nbd $V \subseteq S$ and a local isometry

$$\varphi : V \to S'$$

and $\forall p' \in S'$, $\exists$ nbd $V' \subseteq S'$ and a local isometry

$$\varphi : V' \to S$$

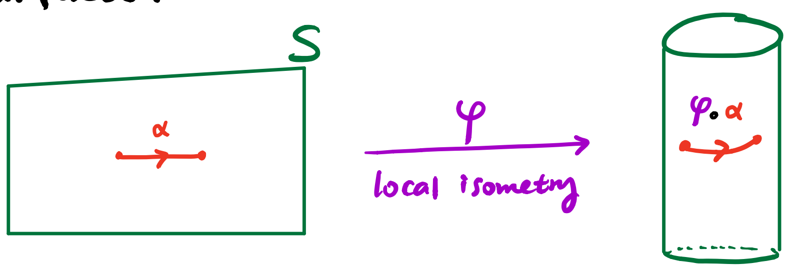
Remark: Two surfaces can be locally isometric without being isometric, for example,



Ex: Show that the helicoid and catenoid are locally isometric.

Basically, "intrinsic geometry" is the study of properties / quantities that are invariant under isometries!

Prop: Local isometries preserve the length of curves on surfaces.



Proof:

$$\underbrace{\int_a^b \| \alpha'(t) \| \, dt}_{\text{Length}(\alpha)} = \underbrace{\int_a^b \| d\varphi_{\alpha(t)}(\alpha'(t)) \| \, dt}_{\text{Length}(\varphi \circ \alpha)}$$

$\square$

Ex: Area is preserved under isometries.

Note: Since a plane has $H \equiv 0$ but $H \neq 0$ for cylinders, the mean curvature H is NOT intrinsic.

However, we will see later that the Gauss curvature K is actually intrinsic!

§ Calculus of vector fields in $\mathbb{R}^n$

Def$^{\underline{n}}$: (vector fields as directional derivatives)

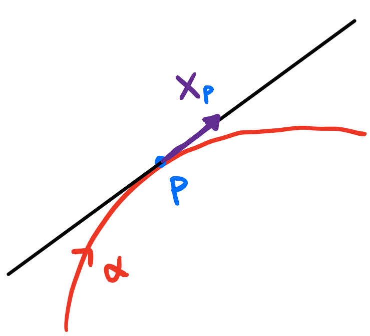
Given a vector field $X : \mathbb{R}^n \to \mathbb{R}^n$, it defines an operator on smooth functions on $\mathbb{R}^n$:

$$\begin{aligned} X \ : \ C^\infty(\mathbb{R}^n) &\longrightarrow C^\infty(\mathbb{R}^n) \\ f &\longmapsto X(f) \end{aligned}$$

where $X(f)(p) := D_{X_p} f(p)$ directional derivative of f at p along X_p

$$= \frac{d}{dt}\Big|_{t=0} f(p + t X_p)$$

$$= \frac{d}{dt}\Big|_{t=0} f(\alpha(t))$$ for ANY curve α s.t. $\alpha(0) = p$, $\alpha'(0) = X_p$



Properties: (1) Linearity: $X(af+bg) = aX(f) + bX(g)$

(2) Leibniz rule: $X(fg) = fX(g) + gX(f)$

where $a, b \in \mathbb{R}$ are constants, $f, g \in C^\infty(\mathbb{R}^n)$.

Note: Given a vector field X and $f \in C^\infty(\mathbb{R}^n)$, one can define a new vector field fX s.t. $(fX)(p) = f(p)\,X(p)$.

Properties: (1) Linearity: $(aX + bY)(f) = aX(f) + bX(g)$

(2) Tensorial: $(fX)(g) = f(X(g))$

In terms of the Euclidean coordinates $x^1, \dots, x^n$ on $\mathbb{R}^n$, we can express any vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ as

$$X(x^1, \dots, x^n) = \underbrace{(a^1(x^1, \dots, x^n), \dots\dots, a^n(x^1, \dots, x^n))}_{\text{smooth functions}}$$

$$= \sum_{i=1}^{n} a^i e_i \quad , \quad \{e_i\} \text{ std. basis of } \mathbb{R}^n$$

Since e_i corresponds to $\frac{\partial}{\partial x^i}$ as operators, therefore

$$\boxed{X = a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + \cdots + a^n \frac{\partial}{\partial x^n}}$$

Since vector fields can be viewed as operators on $C^\infty(\mathbb{R}^n)$, we can consider their compositions:

$$C^\infty(\mathbb{R}^n) \underset{Y}{\overset{X}{\rightrightarrows}} C^\infty(\mathbb{R}^n) \underset{X}{\overset{Y}{\rightrightarrows}} C^\infty(\mathbb{R}^n)$$

i.e. $f \longmapsto X(f) \longmapsto Y(X(f))$

or $f \longmapsto Y(f) \longmapsto X(Y(f))$

Question: Does $X(Y(f)) = Y(X(f))$?

Yes for $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ since mixed partial derivatives commute: $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$

No in general.

Def$^{\underline{n}}$: For any vector fields X, Y on $\mathbb{R}^n$, we define their Lie bracket as

$$[X, Y] := XY - YX$$

i.e. $[X,Y](f) = X(Y(f)) - Y(X(f))$.

$$\forall f \in C^\infty(\mathbb{R}^n)$$

Note: $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] \equiv 0$

Lemma: $[X,Y]$ is a vector field.

Proof: Write $X = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$; $Y = \sum_{j=1}^n b^j \frac{\partial}{\partial x^j}$

$$X(Y(f)) = \Big(\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\Big)\Big(\sum_{j=1}^n b^j \frac{\partial f}{\partial x^j}\Big)$$

$$= \sum_{i,j=1}^n a^i b^j \boxed{\frac{\partial^2 f}{\partial x^i \partial x^j}} + \sum_{i,j=1}^n a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j}$$

$$\parallel$$

Similarly,

$$Y(X(f)) = \sum_{i,j=1}^n a^i b^j \boxed{\frac{\partial^2 f}{\partial x^j \partial x^i}} + \sum_{i,j=1}^n b^j \frac{\partial a^i}{\partial x^j} \frac{\partial f}{\partial x^i}$$

$$\Rightarrow [X,Y](f) = \sum_{i,j=1}^n a^i \frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} - \sum_{i,j=1}^n b^j \frac{\partial a^i}{\partial x^j} \frac{\partial f}{\partial x^i}$$

i.e. $$[X,Y] = \underbrace{\sum_{i,j=1}^n \Big(a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i}\Big) \frac{\partial}{\partial x^j}}_{\text{a vector field !}}$$

$\square$

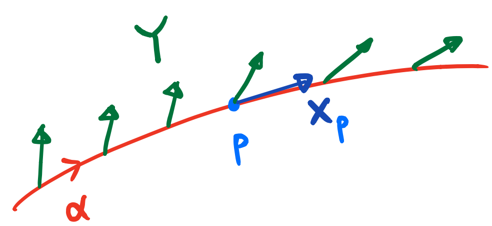
Now we recall how to take directional derivatives of a vector field in $\mathbb{R}^n$, one natural way is to differentiate component-wise:

$$D_X Y = D_X\Big(\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\Big) := \sum_{i=1}^n X(a^i) \frac{\partial}{\partial x^i}.$$

Equivalently, fix a point $p \in \mathbb{R}^n$, to compute $D_X Y(p)$, we take ANY curve α s.t. $\alpha(0) = p$ and $\alpha'(0) = X_p$

$$D_X Y(p) = \frac{d}{dt}\Big|_{t=0} Y(\alpha(t))$$

$$= \lim_{t \to 0} \frac{Y(\alpha(t)) - Y(p)}{t}$$



<u>Properties of $D_X Y$</u>: Let X, Y, Z be vector fields on $\mathbb{R}^n$, $a, b \in \mathbb{R}$ be constants, $f \in C^\infty(\mathbb{R}^n)$. We have:

(1) Linearity in both variables:

$$D_X(aY + bZ) = a D_X Y + b D_X Z$$

$$D_{aX+bY} Z = a D_X Z + b D_Y Z$$

(2) Liebniz rule: $D_X(fY) = X(f)\,Y + f D_X Y$

(3) Tensorial: $D_{fX} Y = f D_X Y$

(4) Torsion free: $\boxed{D_X Y - D_Y X = [X, Y]}$

(5) Metric compatibility:

$$\boxed{X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle X, D_X Z\rangle}$$

Proof: Exercises.